

2008 BLUE MOP, INEQUALITIES-III
ALİ GÜREL

(1) (Korea-98) Let $x, y, z > 0$ with $x + y + z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

(2) (MOP-02) For positive numbers a, b, c , prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

(3) Let $a, b, c > 0$ satisfying $abc = 1$. Prove that

$$\frac{1}{\sqrt{b+\frac{1}{a}+\frac{1}{2}}} + \frac{1}{\sqrt{c+\frac{1}{b}+\frac{1}{2}}} + \frac{1}{\sqrt{a+\frac{1}{c}+\frac{1}{2}}} \geq \sqrt{2}.$$

(4) Let $a, b, c > 0$ satisfying $a + b + c = 1$. Show that

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{\sqrt{abc}}{c+ab} \leq 1 + \frac{3\sqrt{3}}{4}.$$

(5) (KMO-01) Prove that for all $a, b, c > 0$,

$$\sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}.$$

(6) (APMO-05) Let $a, b, c > 0$ such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}.$$

(7) (IMO-01) Let a, b, c be positive numbers. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

(8) (Latvia-02) Let a, b, c, d be positive real numbers such that

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1$$

Prove that $abcd \geq 3$.

Problem 1, Solution by Sam Keller: Let $x = \tan A$, $y = \tan B$, and $c = \tan C$. Because $x + y + z = xyz$, A, B, C are three angles of a triangle. Also because $x, y, z > 0$ the triangle is acute. Then since $\frac{1}{\sqrt{1+x^2}} = \cos A$, we want $\cos A + \cos B + \cos C \leq \frac{3}{2}$. Because $x \rightarrow \cos x$ is concave on $[0, \pi/2)$, by Jensen's inequality

$$\cos A + \cos B + \cos C \leq 3 \cos \left(\frac{A+B+C}{3} \right) = 3 \cos \left(\frac{\pi}{3} \right) = \frac{3}{2} \quad \square$$

Problem 2, Solution by Zhifan Zhang: We claim that $\left(\frac{2a}{b+c} \right)^{\frac{2}{3}} \geq \frac{3a}{a+b+c}$. Observe that the cyclic sum of this inequality gives the desired inequality. Since this inequality is homogenous, after normalization we may assume that $a+b+c=1$ and $0 < a < 1$. Then after cubing both sides and simplifying it becomes equivalent to $a(1-a)^2 \leq \frac{4}{27}$, which follows from AM-GM on the three numbers $2a$, $1-a$ and $1-a$ \square

Problem 3, Solution by Taylor Han: Note that $x+y+1 > \sqrt{2x+2y+1}$. Adding a cyclic sum of these, and using that $abc=1$, we get

$$\begin{aligned} \sum_{cyc} \frac{1}{\sqrt{b+\frac{1}{a}+\frac{1}{2}}} &> \sqrt{2} \left(\frac{1}{b+\frac{1}{a}+1} + \frac{1}{c+\frac{1}{b}+1} + \frac{1}{a+\frac{1}{c}+1} \right) \\ &= \sqrt{2} \left(\frac{1}{b+\frac{1}{a}+1} + \frac{b}{\frac{1}{a}+1+b} + \frac{\frac{1}{a}}{1+b+\frac{1}{a}} \right) = \sqrt{2} \quad \square \end{aligned}$$

Problem 4, Solution by Wenyu Cao: By AM-GM, $\frac{ab+3c}{2} \geq \sqrt{3abc}$. Thus,

$$\frac{\sqrt{abc}}{c+ab} \leq \frac{\sqrt{3}(ab+3c)}{6(ab+c)} = \frac{\sqrt{3}}{6} + \frac{\sqrt{3}c}{3(c+ab)}.$$

Furthermore,

$$\frac{a}{a+bc} + \frac{b}{b+bc} = \frac{a(b+c)+b(a+c)}{(a+b)(b+c)(c+a)} = \frac{c+\frac{2ab}{a+b}}{c+ab} = \frac{c-\frac{2c}{1-c}}{c+ab} + \frac{2}{1-c}.$$

Thus, it suffices to show that

$$\frac{\left(1+\frac{\sqrt{3}}{3}\right)c-\frac{2c}{1-c}}{c+ab} + \frac{2}{1-c} \leq 1 + \frac{7\sqrt{3}}{12}.$$

Note that $\left(1+\frac{\sqrt{3}}{3}\right)c-\frac{2c}{1-c} \leq 0$ for all $0 \leq c \leq 1$. Thus, by AM-GM we get

$$\frac{\left(1+\frac{\sqrt{3}}{3}\right)c-\frac{2c}{1-c}}{c+ab} + \frac{2}{1-c} \leq \frac{\left(1+\frac{\sqrt{3}}{3}\right)c-\frac{2c}{1-c}}{c+\left(\frac{1-c}{2}\right)^2} + \frac{2}{1-c}.$$

Let $f(c)$ denote the RHS of the last inequality. After some calculations we find that, $f'(7-4\sqrt{3})=0$ and investigating the sign of f' we see that the maximum value of f is $f(7-4\sqrt{3})=1+\frac{7\sqrt{3}}{12}$. Hence, we are done \square

Problem 5, Solution by Damien Jiang: Dividing both sides by abc and letting $x = \frac{a}{b}$, $y = \frac{b}{c}$, and $z = \frac{c}{a}$ we get the equivalent inequality:

$$\sqrt{(xy + yz + zx)(x + y + z)} \geq 1 + \sqrt[3]{(x + y)(y + z)(z + x)}, \text{ where } x, y, z > 0, \text{ and } xyz = 1.$$

Let $A = (x + y)(y + z)(z + x)$. Note that $(xy + yz + zx)(x + y + z) = A + xyz = A + 1$. So it remains to prove that $\sqrt{A + 1} \geq 1 + \sqrt[3]{A}$ which after squaring and simplifying becomes equivalent to $(\sqrt[3]{A} - 2)(\sqrt[3]{A} + 1) \geq 0$. Finally this follows from AM-GM, i.e. $A \geq 8xyz = 8$ or $\sqrt[3]{A} \geq 2 \square$

Problem 6, Solution by Damien Jiang (finally): First note that $\frac{1}{\sqrt[3]{1+x^3}} \geq \frac{2}{2+x^2}$. Using this and letting $x = \frac{a^2}{4}, y = \frac{b^2}{4}, z = \frac{c^2}{4}$, it suffices to prove the following inequality:

$$\frac{x}{(1+2x)(1+2y)} + \frac{y}{(1+2y)(1+2z)} + \frac{z}{(1+2z)(1+2x)} \geq \frac{1}{3}$$

where $x, y, z > 0$ such that $xyz = 1$. After multiplying out and canceling terms, we get the equivalent inequality

$$(x + y + z) + 2(xy + yz + zx) \geq 9,$$

which follows from AM-GM, since $xyz = 1 \square$

Problem 7, Solution by Minseon Shin: By weighted Jensen on the convex function $f(x) = \frac{1}{\sqrt{x}}$, we have

$$\frac{\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ac}} + \frac{c}{\sqrt{c^2+8ab}}}{a+b+c} \geq \frac{1}{\sqrt{\frac{a(a^2+8bc)+b(b^2+8ca)+c(c^2+8ab)}{a+b+c}}}.$$

We need to show that

$$\frac{a+b+c}{\sqrt{\frac{a^3+b^3+c^3+24abc}{a+b+c}}} \geq 1,$$

which is equivalent to

$$(a+b+c)^3 \geq a^3 + b^3 + c^3 + 24abc \Leftrightarrow \sum_{sym} a^2b \geq \sum_{sym} abc$$

which is Muirhead inequality: $[2, 1, 0] \geq [1, 1, 1] \square$

Problem 8, Solution by Joshua Pfeffer: Let $\log S = \log a + \log b + \log c + \log d < \log 3$. Observe that the function $x \rightarrow \frac{1}{1+e^{4x}}$ is convex. So by Jensen's Inequality:

$$1 = \sum_{cyc} \frac{1}{1+a^4} = \sum_{cyc} \frac{1}{1+e^{4\log a}} \geq \frac{4}{1+e^{\log S}},$$

which implies that $S \geq 3$, as desired \square